QUASI INVARIANT STOCHASTIC FLOWS OF SDES WITH NON-SMOOTH DRIFTS ON RIEMANNIAN MANIFOLDS*

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Abstract. In this article we prove that stochastic differential equation (SDE) with Sobolev drift on compact Riemannian manifold admits a unique ν-almost everywhere stochastic invertible flow, where ν is the Riemannian measure, which is quasi-invariant with respect to ν. In particular, we extend the well known DiPerna-Lions flows of ODEs to SDEs on Riemannian manifold.

1. Introduction

Let M be a connected and compact C^∞-manifold of dimension d. Consider the following Stratonovich's stochastic differential equation (SDE) on M:

$$\mathrm{d}x_t = X_0(x_t)\mathrm{d}t + X_k(x_t) \circ \mathrm{d}W_t^k, \quad x_0 = x, \tag{1.1}$$

where $X_i, i = 0, \cdots, m$ are $m + 1$-vector fields on M, and $(W_t)_{t\geqslant 0}$ is the m-dimensional standard Brownian motion on the classical Wiener space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$, i.e., Ω is the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^m$ with locally uniform convergence topology, $\mathcal{F}$ is the Borel σ-field, P is the Wiener measure, $(\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration generated by the coordinate process $W_t(\omega) = \omega(t)$. Here and below, we use the following convention: if an index appears twice in a product, it will be summed.

For solving SDE (1.1), there are usually two ways: One way is to first construct the solutions in local coordinates and then patches up them (cf. [8]). Another way is that one embeds M into some Euclidean space, obtains a solution in this larger space, and then proves that the solution will actually stay in M if the starting point x is in M (cf. [7]). Both of these arguments require that X_k, $k = 0, \cdots, m$ are smooth (at least C^2) vector fields.

In the case of flat Euclidean space, a celebrated theory established by DiPerna and Lions [5] says that when X_0 only has Sobolev regularity and bounded divergence, ODE

$$\mathrm{d}x_t = X_0(x_t)\mathrm{d}t, x_0 = x$$

defines a unique regular Lagrangian flow in the sense of Lebesgue measure. Their proofs are based on a new notion called renormalized solution for the associated transport equation:

$$\partial_t u + X_0 u = \partial_t u + X_0^i \partial_i u = 0, \quad u|_{t=0} = u_0,$$

where X_0^i is the component of vector field X_0 under natural frames. For the DiPerna-Lions flow on compact Riemannian manifold, Dumas, Golse and Lochak [6] gave an outline for the proof.

Recently, we have extended DiPerna-Lions' flow to the case of SDEs in [16]. Therein, we followed the direct argument of Crippa and De Lellis [4]. It is worth pointing out that we can not use the original method of DiPerna and Lions to study the SDEs with Sobolev drifts because the associated stochastic partial differential equation is always degenerate (cf. [16]). On the other

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hand, when we consider the corresponding SDEs with Sobolev drifts on Riemannian manifold, it seems that we can not use the localizing and patching method as well as the embedding method since X_0 is not smooth and the solution is only defined for almost all starting points. In order to extend the result in [16] to Riemannian manifold, we shall directly use the intrinsic Riemannian distance as in [13]. For this aim, we have to make a detailed analysis for the distance function associated with the Riemannian metric.

This paper is organized as follows: in Section 2, we give the notion of ν-almost everywhere stochastic flow of SDE (1.1) and state our main result. In Section 3, we analyze the distance function on Riemannian manifold and give some necessary preliminaries. In Section 4, we prove our main result as in [4] and [16] by using the Hardy-Littlewood maximal function on Riemannian manifold.

2. Main Result

Let $(M, \mathfrak{g})$ be a connected and compact C^∞ Riemannian manifold of d-dimension, where $\mathfrak{g}$ denotes the Riemannian metric, a symmetric, positively definite, and second order covariant tensor field on M. Let $\nu(\mathrm{d}x)$ be the Riemannian measure, and ∇ the Levi-Civita connection associated with $\mathfrak{g}$. We also use ∇ to denote the gradient operator. The divergence operator denoted by div is the dual operator of ∇ with respect to ν. Let TM be the tangent bundle. For any $x \in M$, the length of a vector $X \in T_xM$ is denoted by $|X|_x := \sqrt{\mathfrak{g}_x(X, X)}$. Letting $\mathcal{T}$ be a measurable transformation of M, we use $\nu \circ \mathcal{T}$ to denote the image measure of ν under $\mathcal{T}$, i.e., for any nonnegative measurable function f,

$$\int_M f(x)\nu \circ \mathcal{T}(\mathrm{d}x) = \int_M f(\mathcal{T}(x))\nu(\mathrm{d}x).$$

By $\nu \circ \mathcal{T} \ll \nu$, we mean that $\nu \circ \mathcal{T}$ is absolutely continuous with respect to ν.

We first introduce the following notion of ν-almost everywhere stochastic (invertible) flows (cf. [11] [1] [16]).

Definition 2.1. *Let $x_t(\omega, x)$ be an M-valued measurable stochastic field on $\mathbb{R}_+ \times \Omega \times M$. We say $x_t(x)$ a ν-**almost everywhere stochastic flow** of (1.1) corresponding to vector fields $(X_k)_{k=0,\cdots,m}$ if*

(A) *For ν-almost all $x \in \mathbb{R}^d$, $t \mapsto x_t(x)$ is a continuous and $(\mathcal{F}_t)$-adapted stochastic process and, satisfies that for any $T > 0$ and $f \in C^\infty(M)$,*

$$f(x_t(x)) = f(x) + \int_0^t X_0 f(x_s(x))\mathrm{d}s + \int_0^t X_k f(x_s(x)) \circ \mathrm{d}W_s^k, \quad \forall t \geqslant 0.$$

(B) *For any $t \geqslant 0$ and P-almost all $\omega \in \Omega$, $\nu \circ x_t(\omega, \cdot) \ll \nu$. Moreover, for any $T > 0$, there exists a constant $K_{T,X_0,X_k} > 0$ such that for all nonnegative measurable function f on M,*

$$\sup_{t\in[0,T]} \mathbb{E} \int_M f(x_t(x))\nu(\mathrm{d}x) \leqslant K_{T,X_0,X_k} \int_M f(x)\nu(\mathrm{d}x). \tag{2.1}$$

*We say $x_t(x)$ a ν-**almost everywhere stochastic invertible flow** of (1.1) corresponding to vector fields $(X_k)_{k=0,\cdots,m}$ if in addition to the above* **(A)** *and* **(B)**,

(C) *For any $t \geqslant 0$ and P-almost all $\omega \in \Omega$, there exists a measurable inverse $x_t^{-1}(\omega, \cdot)$ of $x_t(\omega, \cdot)$ so that $\nu \circ x_t^{-1}(\omega, \cdot) = \rho_t(\omega, \cdot)\nu$, where the density $\rho_t(x)$ is given by*

$$\rho_t(x) := \exp\left\{\int_0^t \mathrm{div}X_0(x_s(x))\mathrm{d}s + \int_0^t \mathrm{div}X_k(x_s(x)) \circ \mathrm{d}W_s^k\right\}. \tag{2.2}$$

Remark 2.2. *In the above definitions, we have already assumed that all the integrals make sense. In particular, the above property* **(C)** *guarantees the quasi invariance of the flow transformation $x \mapsto x_t(x)$ with respect to the Riemannian volume.*

For $k \in \mathbb{N} \cup \{\infty\}$, let $C^k(TM)$ be the set of all k-order smooth vector fields on M. For $p \geqslant 1$ and $X \in C^\infty(TM)$, we define

$$\|X\|_p := \left(\int_M |X|_x^p \nu(\mathrm{d}x)\right)^{1/p}$$

and

$$\|X\|_{1,p} := \|X\|_p + \left(\int_M |\nabla X|_x^p \nu(\mathrm{d}x)\right)^{1/p}.$$

Let $L^p(TM)$ and $\mathbb{H}_1^p(TM)$ be the completions of $C^\infty(TM)$ with respect to $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ respectively. We also use $L^\infty(TM)$ to denote the set of all bounded measurable vector fields.

The following two propositions are direct consequences of Definition 2.1, whose proofs can be found in [16].

Proposition 2.3. *Assume that SDE (1.1) admits a unique ν-almost everywhere stochastic flow. Then the following flow property holds: for any $s \geqslant 0$ and $(P \times \nu)$-almost all $(\omega, x) \in \Omega \times M$,*

$$x_{t+s}(\omega, x) = x_t(\theta_s\omega, x_s(\omega, x)), \ \forall t \geqslant 0,$$

where $\theta_s\omega := \omega(s + \cdot) - \omega(s)$. Moreover, for any bounded measurable function f on M, define

$$\mathbb{T}_t f(x) := \mathbb{E} f(x_t(x)),$$

then for any $t, s \geqslant 0$

$$\mathbb{E}(f(x_{t+s}(x))|\mathcal{F}_s) = \mathbb{T}_t f(x_s(x)), \ \ (P \times \nu) - a.e.$$

In particular, $(\mathbb{T}_t)_{t\geqslant 0}$ forms a bounded linear operator semigroup on $L^p(M)$ for any $p \geqslant 1$.

Proposition 2.4. *Assume that $X_0 \in L^\infty(TM)$ with $\mathrm{div}X_0 \in L^1(M)$ and $X_k \in C^2(TM)$, $k = 1, \cdots, m$. Let $x_t(x)$ be a ν-almost everywhere stochastic invertible flow of (1.1) in the sense of Definition 2.1. Let $u_0 \in L^1(M)$ and set $u_t(x) := u_0(x_t^{-1}(x))$. Then $u_t(x)$ solves the following stochastic transport equation in the distributional sense:*

$$\mathrm{d}u = -X_0 u \mathrm{d}t - X_k u \circ \mathrm{d}W_t^k.$$

In particular, $\bar{u}_t(x) := \mathbb{E}u_0(x_t^{-1}(x))$ is a distributional solution of the following second order parabolic differential equation:

$$\partial_t \bar{u} = -\frac{1}{2}\sum_k X_k^2 \bar{u} - X_0 \bar{u}.$$

Our main result in the present paper is:

Theorem 2.5. *Assume that $X_0 \in \mathbb{H}_1^p(TM) \cap L^\infty(TM)$ for some $p > 1$, satisfies*

$$\mathrm{div}X_0 \in L^\infty(M),$$

and for each $k = 1, \cdots, m$, $X_k \in C^2(TM)$. Then there exists a unique ν-almost everywhere stochastic invertible flows $\{x_t(x), x \in M\}_{t\geqslant 0}$ of SDE (1.1) in the sense of Definition 2.1.

3. Preliminaries

3.1. **Distance Function.** We need the following simple lemma.

Lemma 3.1. *Let $(M, \mathbf{d})$ be a compact metric space. Let $\Sigma = \{U_\alpha, \alpha \in \Lambda\}$ be a finite open covering of M. Then there exists a positive number ϱ such that for any $x, y \in M$, if $\mathbf{d}(x, y) < \varrho$, then x, y must lies in some U_α simultaneously.*

Proof. We use the contradiction method. Suppose that for any $n \in \mathbb{N}$, there exists $x_n, y_n \in M$ with $\mathbf{d}(x_n, y_n) < \frac{1}{n}$ such that

$$x_n, y_n \text{ do not belong to any } U_\alpha \in \Sigma \text{ simultaneously.} \tag{3.1}$$

By the compactness of M, there is a subsequence n_k and $z \in M$ such that

$$\lim_{k\to\infty} \mathbf{d}(x_{n_k}, z) = 0, \quad \lim_{k\to\infty} \mathbf{d}(y_{n_k}, z) = 0.$$

Since z belongs to some open set $U_\alpha \in \Sigma$, for k large enough, x_{n_k} and y_{n_k} must lies in U_α, which is contrary to (3.1). The proof is complete. □

Using this lemma, we have the following property about the distance function $\mathbf{d}(\cdot, \cdot)$ on M, which will be our localizing basis below.

Lemma 3.2. *Let M be a compact Riemannian manifold. Then, there are a finite covering $\{(U_\alpha, \varphi_\alpha; \xi^k_\alpha)\}_{\alpha\in\Lambda}$ of M by normal coordinate neighborhoods, and positive constants $\varrho, \lambda \in (0, 1)$ such that*

(1^o) For any $x, y \in M$ with $\mathbf{d}(x, y) < \varrho$, x, y must be in some U_α simultaneously, and there is a unique minimizing geodesic connecting x and y in U_α.

(2^o) In local coordinate $\{(U_\alpha, \varphi_\alpha; \xi^k_\alpha\}$, for any $x, y \in U_\alpha$,

$$\lambda \cdot |\varphi_\alpha(x) - \varphi_\alpha(y)| \leqslant \mathbf{d}(x, y) \leqslant \lambda^{-1} \cdot |\varphi_\alpha(x) - \varphi_\alpha(y)|,$$

where $|\cdot|$ denotes the Euclidean metric in $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^d$. Moreover,

$$\lambda I \leqslant (g^\alpha_{ij}) \leqslant \lambda^{-1} I,$$

where $g^\alpha_{ij} := \mathrm{g}(\partial_{\xi^i_\alpha}, \partial_{\xi^j_\alpha})$.

(3^o) For any U_α, the restriction of $\mathbf{d}^2(\cdot, \cdot)$ to $U_\alpha \times U_\alpha$ belongs to $C^\infty(U_\alpha \times U_\alpha)$.

Proof. For each $a \in M$, there is a normal coordinate neighborhood (U_a, φ_a) of a such that any two points in U_a can be joined by a unique minimizing geodesic lying in U_a, and $\mathbf{d}^2(\cdot, \cdot) \in C^\infty(U_a \times U_a)$ (see [9, p.166, Theorem 3.6]). Moreover, there is a constant λ_a such that for all $x, y \in U_a$ (see [3, p.125]),

$$\lambda_a^{-1} |\varphi_a(x) - \varphi_a(y)| \leqslant \mathbf{d}(x, y) \leqslant \lambda_a |\varphi_a(x) - \varphi_a(y)|$$

and

$$\lambda_a^{-1} I \leqslant (g^\alpha_{ij}) \leqslant \lambda_a I,$$

The results now follow by the compactness of M and Lemma 3.1. □

In the following, we shall fix the $\Sigma := \{(U_\alpha, \varphi_\alpha; \xi^k_\alpha)\}_{\alpha\in\Lambda}$ and ϱ, λ in this lemma as well as a unit partition $(\psi_\alpha)_{\alpha\in\Lambda}$ subordinate to Σ, i.e.,

$$\psi_\alpha \in C^\infty(M; [0, 1]), \quad \mathrm{supp}(\psi_\alpha) \subset U_\alpha, \quad \sum_{\alpha\in\Lambda} \psi_\alpha \equiv 1. \tag{3.2}$$

Given two points $x, y \in M$ with $\mathbf{d}(x, y) < \varrho$, let

$$\{\gamma(s), \ s \in [0, t_0], \ t_0 := \mathbf{d}(x, y)\}$$

with

$$\gamma(0) = x, \ \ \gamma(t_0) = y$$

be the unique minimizing geodesic connecting x and y. We use $//^{\gamma}_{y\leftarrow x}$ to denote the parallel transport from x to y along the geodesic γ, i.e., $//^{\gamma}_{y\leftarrow x}$ establishes an isomorphism between tangent spaces T_xM and T_yM. For a vector field X and a smooth function f, we write

$$\mathfrak{g}_x(X, \nabla f) = X(x)f = [Xf](x).$$

Lemma 3.3. *For $x \in M$ and a vector $X \in T_xM$, we have*

$$\mathfrak{g}_x(X, \nabla \mathbf{d}(\cdot, y)) = -\mathfrak{g}_y(//^{\gamma}_{y\leftarrow x}X, \nabla \mathbf{d}(x, \cdot)).$$

Proof. By a corollary to Gaussian Lemma (see e.g. [12, Corollary 6.9]), we have

$$\begin{aligned} \mathfrak{g}_x(X, \nabla \mathbf{d}(\cdot, y)) &= \mathfrak{g}_x(X, \dot\gamma(0)) = -\mathfrak{g}_y(//^{\gamma}_{y\leftarrow x}X, \dot\gamma(t_0)) \\ &= -\mathfrak{g}_y(//^{\gamma}_{y\leftarrow x}X, \nabla \mathbf{d}(x, \cdot)). \end{aligned}$$

□

3.2. **Local maximal function on Riemannian manifold M.** Convention: For two expressions A and B, the notation $A \preceq B$ means that $A \leqslant C \cdot B$, where $C > 0$ is an unimportant constant and may change in different occasions. We assume that the reader can see the dependence of C on the parameters from the context.

For a nonnegative function $f \in L^1(M)$ and $R > 0$, the local maximal function $\mathcal{M}_R f$ is defined by

$$\mathcal{M}_R f(x) := \sup_{r\in(0,R)} \frac{1}{\nu(B_r(x))} \int_{B_r(x)} f(y)\nu(\mathrm{d}y),$$

where $B_r(x) := \{y \in M : \mathbf{d}(x, y) < r\}$. Similarly, for a function $h \in L^1_{loc}(\mathbb{R}^d)$, we define the local maximal function $\tilde{\mathcal{M}}_R h$ in Euclidean space $\mathbb{R}^d$ by

$$\tilde{\mathcal{M}}_R h(\xi) := \sup_{r\in(0,R)} \frac{1}{|\tilde{B}_r(\xi)|} \int_{\tilde{B}_r(\xi)} h(\eta)\mathrm{d}\eta,$$

where $\tilde{B}_r(\xi) := \{\eta \in \mathbb{R}^d : |\eta - \xi| < r\}$ and $|\tilde{B}_r(\xi)|$ denotes the volume of ball $\tilde{B}_r(\xi)$ with respect to the Lebesgue measure.

We have

Lemma 3.4. *Let f be a measurable function on M with $\nabla f \in L^1(TM)$. Then, there exists a ν-null set N such that for all $x, y \notin N$ with $\mathbf{d}(x, y) < \lambda^2\varrho$,*

$$|f(x) - f(y)| \preceq \mathbf{d}(x, y) \cdot (\mathcal{M}_\varrho|\nabla f|(x) + \mathcal{M}_\varrho|\nabla f|(y)),$$

where λ and ϱ are from Lemma 3.2.

Proof. Since $\mathbf{d}(x, y) < \varrho$, by ($1^o$) of Lemma 3.2 we only need to prove the lemma in local coordinate $(U, \varphi; \xi^k) \in \Sigma$. It is well known that there is a Lebesgue-null set Q such that for all $\xi, \eta \in \varphi(U) \setminus Q$ with $|\xi - \eta| < \lambda\varrho$ (cf. [4, Appendix]),

$$|f \circ \varphi^{-1}(\xi) - f \circ \varphi^{-1}(\eta)| \preceq |\xi - \eta| \cdot (\tilde{\mathcal{M}}_{\lambda\varrho}|\nabla(f \circ \varphi^{-1})|(\xi) + \tilde{\mathcal{M}}_{\lambda\varrho}|\nabla(f \circ \varphi^{-1})|(\eta)).$$

Noting that by (2^o) of Lemma 3.2,

$$\varphi(B_{\lambda r}(x)) \subset \tilde{B}_r(\varphi(x)) \subset \varphi(B_{\lambda^{-1}r}(x))$$

and

$$\lambda^{d/2}\nu(B_{\lambda r}(x)) \leqslant |\tilde{B}_r(\varphi(x))| \leqslant \lambda^{-d/2}\nu(B_{\lambda^{-1}r}(x)),$$

we thus have

$$\tilde{\mathcal{M}}_{\lambda\varrho}|\nabla(f \circ \varphi^{-1})|(\xi) \preceq \mathcal{M}_\varrho|\nabla f|(\varphi^{-1}(\xi)).$$

The result now follows. $\square$

The following result can be proved along the same lines as in [14, p.5 Theorem 1].

Lemma 3.5. *Let $f \in L^p(M)$ for some $p > 1$, then*

$$\|\mathcal{M}_R f\|_p \preceq \|f\|_p. \tag{3.3}$$

3.3. **Two estimates about vector fields.**

Lemma 3.6. *Let $X \in \mathbb{H}^1_1(TM)$ be a Sobolev vector field. Then there exists a ν-null set N such that for all $x, y \notin N$ with $\mathbf{d}(x,y) < \lambda^2\varrho$,*

$$|X(x)\mathbf{d}^2(\cdot, y) + X(y)\mathbf{d}^2(x,\cdot)| \preceq \mathbf{d}^2(x,y)\cdot(1 + \mathcal{M}_\varrho|X|_1(x) + \mathcal{M}_\varrho|X|_1(y)), \tag{3.4}$$

where $|X|_1(x) := |X|_x + |\nabla X|_x$, and the constant in $\preceq$ is independent of X. In particular, if $X \in C^1(TM)$, then

$$|X(x)\mathbf{d}^2(\cdot, y) + X(y)\mathbf{d}^2(x,\cdot)| \preceq \mathbf{d}^2(x,y)\cdot\Big(2\sup_{x\in M}|X|_1(x) + 1\Big). \tag{3.5}$$

Proof. By Lemma 3.3, we have

$$\begin{aligned} |X(x)\mathbf{d}^2(\cdot, y) + X(y)\mathbf{d}^2(x,\cdot)| &= 2\mathbf{d}(x,y)\cdot|\mathrm{g}_y(X(y) - //^\gamma_{y\leftarrow x}X(x), \nabla\mathbf{d}(x,\cdot))| \\ &\leqslant 2\mathbf{d}(x,y)\cdot|X(y) - //^\gamma_{y\leftarrow x}X(x)|_y. \end{aligned}$$

Thus, it is enough to prove that there exists a ν-null set N such that for all $x, y \notin N$ with $\mathbf{d}(x,y) < \lambda^2\varrho$,

$$|X(y) - //^\gamma_{y\leftarrow x}X(x)|_y \preceq \mathbf{d}(x,y)\cdot(1 + \mathcal{M}_\varrho|X|_1(x) + \mathcal{M}_\varrho|X|_1(y)).$$

Since $\mathbf{d}(x,y) < \lambda^2\varrho$, we only need to prove it in a local coordinate $(U, \varphi; \xi^k) \in \Sigma$. In local coordinate $(U, \varphi; \xi^k)$, we may write

$$X(x)|_U = X^k(x)\partial_{\xi^k}$$

and

$$\nabla X(x)|_U = (\partial_{\xi^i}X^k + X^j\Gamma^k_{ji})\mathrm{d}\xi^i \otimes \partial_{\xi^k},$$

where $\Gamma^k_{ji} = \mathrm{g}(\nabla_{\partial_{\xi^j}}\partial_{\xi^i}, \partial_{\xi^k})$ are Christoffel symbols. By Lemma 3.4, there exists a ν-null set N such that for all $x, y \notin U \setminus N$ with $\mathbf{d}(x,y) < \lambda^2\varrho$,

$$\begin{aligned} |X^k(x) - X^k(y)| &\preceq \mathbf{d}(x,y)\cdot(\mathcal{M}_\varrho|\nabla X^k|(x) + \mathcal{M}_\varrho|\nabla X^k|(y)) \\ &\preceq \mathbf{d}(x,y)\cdot(\mathcal{M}_\varrho|X|_1(x) + \mathcal{M}_\varrho|X|_1(y)). \end{aligned} \tag{3.6}$$

Let $t_0 := \mathbf{d}(x,y)$, and $\{Y^k_s, s \in [0, t_0], k = 1, \cdots, d\}$ be the unique solution to ODEs

$$\frac{\mathrm{d}Y^k_s}{\mathrm{d}s} + \sum_{ij}\Gamma^k_{ij}(\gamma(s))\cdot Y^i_s\cdot\dot\gamma^j_s = 0, \ \ Y^k_0 = X^k(x), \ k = 1, \cdots, d.$$

Then $//^\gamma_{y\leftarrow x}X(x) = Y^k_{t_0}\cdot\partial_{\xi^k}$. From this equation, one easily finds that

$$|Y^k_{t_0} - X^k(x)| = |Y^k_{t_0} - Y^k_0| \preceq t_0 = \mathbf{d}(x,y). \tag{3.7}$$

Hence, by (2^o) of Lemma 3.2,

$$\begin{aligned} |X(y) - //^\gamma_{y\leftarrow x}X(x)|_y &= \Big((X^k(y) - Y^k_{t_0})\cdot(X^j(y) - Y^j_{t_0})\cdot g_{kj}(y)\Big)^{1/2} \\ &\preceq \sum_{k=1}^d |X^k(y) - Y^k_{t_0}| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{d}\left(|X^k(y)-X^k(x)|+|X^k(x)-Y^k_{t_0}|\right)\\
&\le \mathbf{d}(x,y)\cdot(1+\mathcal{M}_\varrho|X|_1(x)+\mathcal{M}_\varrho|X|_1(y)),
\end{aligned}
$$

where the last step is due to (3.6) and (3.7). The proof is finished. $\square$

Lemma 3.7. *Let X be a C^2-vector field on M. Then for any $x, y \in M$ with $\mathbf{d}(x,y) < \varrho$,*

$$
|(X^2\mathbf{d}^2)_{11}(x,y)+(X^2\mathbf{d}^2)_{12}(x,y)+(X^2\mathbf{d}^2)_{21}(x,y)+(X^2\mathbf{d}^2)_{22}(x,y)| \preceq \mathbf{d}^2(x,y), \tag{3.8}
$$

where $(X^2\mathbf{d}^2)_{12}(x,y) = X(y)X(x)\mathbf{d}^2(x,y)$ and similarly for others, and the constant in $\preceq$ may depend on X.

Proof. First of all, we have

$$
(X^2\mathbf{d}^2)_{11}(x,y) = X\mathfrak{g}(X,\nabla\mathbf{d}^2(\cdot,y))(x) = \mathfrak{g}_x(\nabla_X X,\nabla\mathbf{d}^2(\cdot,y)) + \mathfrak{g}_x(X,\nabla_X\nabla\mathbf{d}^2(\cdot,y))
$$

and

$$
(X^2\mathbf{d}^2)_{22}(x,y) = X\mathfrak{g}(X,\nabla\mathbf{d}^2(x,\cdot))(y) = \mathfrak{g}_y(\nabla_X X,\nabla\mathbf{d}^2(x,\cdot)) + \mathfrak{g}_y(X,\nabla_X\nabla\mathbf{d}^2(x,\cdot)),
$$

where the second equality is due to the property of the Levi-Civia connection.

By Lemma 3.3, we also have

$$
\begin{aligned}
(X^2\mathbf{d}^2)_{12}(x,y) &= X\mathfrak{g}_x(X,\nabla\mathbf{d}^2(\cdot,y))(y) = -X\mathfrak{g}(//^{\gamma}_{\cdot\leftarrow x}X,\nabla\mathbf{d}^2(x,\cdot))(y)\\
&= -\mathfrak{g}_y(\nabla_X(//^{\gamma}_{\cdot\leftarrow x}X),\nabla\mathbf{d}^2(x,\cdot)) - \mathfrak{g}_y(//^{\gamma}_{\cdot\leftarrow x}X,\nabla_X\nabla\mathbf{d}^2(x,\cdot))
\end{aligned}
$$

and

$$
\begin{aligned}
(X^2\mathbf{d}^2)_{21}(x,y) &= X\mathfrak{g}_y(X,\nabla\mathbf{d}^2(x,\cdot))(x) = -X\mathfrak{g}(//^{\gamma}_{\cdot\leftarrow y}X,\nabla\mathbf{d}^2(\cdot,y))(x)\\
&= -\mathfrak{g}_x(\nabla_X(//^{\gamma}_{\cdot\leftarrow y}X),\nabla\mathbf{d}^2(\cdot,y)) - \mathfrak{g}_x(//^{\gamma}_{\cdot\leftarrow y}X,\nabla_X\nabla\mathbf{d}^2(\cdot,y)).
\end{aligned}
$$

Thus,

$$
(X^2\mathbf{d}^2)_{11}(x,y)+(X^2\mathbf{d}^2)_{12}(x,y)+(X^2\mathbf{d}^2)_{21}(x,y)+(X^2\mathbf{d}^2)_{22}(x,y) = \mathrm{I}+\mathrm{II}+\mathrm{III},
$$

where

$$
\begin{aligned}
\mathrm{I} &:= \mathfrak{g}_x(\nabla_X X,\nabla\mathbf{d}^2(\cdot,y)) + \mathfrak{g}_y(\nabla_X X,\nabla\mathbf{d}^2(x,\cdot))\\
&= \mathfrak{g}_x(\nabla_X X - //^{\gamma}_{\cdot\leftarrow y}\nabla_X X,\nabla\mathbf{d}^2(\cdot,y)),\\
\mathrm{II} &:= -\mathfrak{g}_x(\nabla_X(//^{\gamma}_{\cdot\leftarrow y}X),\nabla\mathbf{d}^2(\cdot,y)) - \mathfrak{g}_y(\nabla_X(//^{\gamma}_{\cdot\leftarrow x}X),\nabla\mathbf{d}^2(x,\cdot))\\
&= \mathfrak{g}_x(//^{\gamma}_{x\leftarrow y}(\nabla_X(//^{\gamma}_{\cdot\leftarrow x}X)) - \nabla_X(//^{\gamma}_{\cdot\leftarrow y}X),\nabla\mathbf{d}^2(\cdot,y)),\\
\mathrm{III} &:= \mathfrak{g}_x(X,\nabla_X\nabla\mathbf{d}^2(\cdot,y)) + \mathfrak{g}_y(X,\nabla_X\nabla\mathbf{d}^2(x,\cdot))\\
&\quad -\mathfrak{g}_x(//^{\gamma}_{\cdot\leftarrow y}X,\nabla_X\nabla\mathbf{d}^2(\cdot,y)) - \mathfrak{g}_y(//^{\gamma}_{\cdot\leftarrow x}X,\nabla_X\nabla\mathbf{d}^2(x,\cdot))\\
&= \mathfrak{g}_x(X - //^{\gamma}_{x\leftarrow y}X,\nabla_X\nabla\mathbf{d}^2(\cdot,y) - //^{\gamma}_{x\leftarrow y}\nabla_X\nabla\mathbf{d}^2(x,\cdot)).
\end{aligned}
$$

Now, in a local coordinate $(U,\varphi;\xi^k)\in\Sigma$, set for $k = 1,\cdots,d$

$$
\begin{aligned}
Z^k_1(x,y) &:= (\nabla_X(//^{\gamma}_{\cdot\leftarrow x}X(x)))^k(y),\\
Z^k_2(x,y) &:= (\nabla_X\nabla\mathbf{d}^2(\cdot,y))^k(x).
\end{aligned}
$$

It is easy to see that Z^k_1 and Z^k_2 are C^1 functions on $U\times U$. Hence,

$$
\begin{aligned}
|Z^k_1(x,y)-Z^k_1(y,x)| &\preceq \mathbf{d}(x,y),\\
|Z^k_2(x,y)-Z^k_2(y,x)| &\preceq \mathbf{d}(x,y).
\end{aligned}
$$

As in the proof of Lemma 3.6, one has

$$
|//^{\gamma}_{x\leftarrow y}(\nabla_X(//^{\gamma}_{\cdot\leftarrow x}X)) - \nabla_X(//^{\gamma}_{\cdot\leftarrow y}X)|_x \preceq \mathbf{d}(x,y)
$$

and

$$|\nabla_X \nabla \mathbf{d}^2(\cdot, y) - //^{\gamma}_{x\leftarrow y}(\nabla_X \nabla \mathbf{d}^2(x, \cdot))|_x \leq \mathbf{d}(x, y).$$

Combining (3.5) and the above estimates, we obtain the desired result. □

3.4. **Mollifying a non-smooth vector field.** For any measurable vector field $X \in TM$, recalling (3.2), we may write

$$X = \sum_\alpha \psi_\alpha X = \sum_\alpha \psi_\alpha X|_{U_\alpha} = \sum_\alpha \psi_\alpha X^k_\alpha \partial_{\xi^k_\alpha},$$

where $X^k_\alpha : U_\alpha \to \mathbb{R}$ is the coordinate component of X in local coordinate $(U_\alpha, \varphi_\alpha; \xi^k_\alpha)$.

Let ζ be a nonnegative smooth function on $\mathbb{R}^d$ with support in $\{\xi \in \mathbb{R}^d : |\xi| < 1\}$ and

$$\int_{\mathbb{R}^d} \zeta(\xi)\mathrm{d}\xi = 1.$$

Set $\zeta_n(\xi) := n^d \zeta(n\xi)$ and define

$$X^k_{\alpha,n}(x) := (X^k_\alpha \circ \varphi^{-1}_\alpha * \zeta_n) \circ \varphi_\alpha := \int_{\varphi_\alpha(U_\alpha)} X^k_\alpha \circ \varphi^{-1}_\alpha(\xi) \cdot \zeta_n(\varphi_\alpha(x) - \xi)\mathrm{d}\xi \tag{3.9}$$

and

$$X_n := \sum_\alpha \psi_\alpha X^k_{\alpha,n} \partial_{\xi^k_\alpha}. \tag{3.10}$$

Then it is clear that $X_n \in C^\infty(TM)$.

Remark 3.8. *In general, the restriction of X_n to U_α does not equal to $X^k_{\alpha,n}\partial_{\xi^k_\alpha}$ since for $\alpha \neq \beta$, the following compatibility is not true any more:*

$$X^k_{\alpha,n} \neq X^j_{\beta,n} \partial\xi^k_\alpha / \partial\xi^j_\beta \quad \text{in } U_\alpha \cap U_\beta \neq \emptyset.$$

We have the following proposition.

Proposition 3.9. *Let $X \in \mathbb{H}^p_1(TM)$ for some $p \geqslant 1$ and X_n be defined by (3.10). Then*

$$\lim_{n\to\infty} \|X - X_n\|_{1,p} = 0.$$

Moreover, if $X \in L^\infty(TM)$ satisfies $[\mathrm{div} X]^- \in L^\infty(M)$, then for some constant $C > 0$ independent of n and X,

$$\|[\mathrm{div} X_n]^-\|_{L^\infty(M)} \leqslant C(\|[\mathrm{div} X]^-\|_{L^\infty(M)} + \|X\|_{L^\infty(TM)}). \tag{3.11}$$

Proof. First of all, by (2^o) of Lemma 3.2, we have

$$\begin{aligned}
\lim_{n\to\infty} \|X - X_n\|^p_p &\leqslant \lim_{n\to\infty} \sum_\alpha \int_{U_\alpha} \psi^p_\alpha [(X^k_\alpha - X^k_{\alpha,n})(X^j_\alpha - X^j_{\alpha,n}) g^\alpha_{kj}]^{p/2} \nu(\mathrm{d}x) \\
&\leqslant C \lim_{n\to\infty} \sum_{\alpha,k} \int_{\varphi_\alpha(U_\alpha)} |X^k_\alpha \circ \varphi^{-1}_\alpha - X^k_\alpha \circ \varphi^{-1}_\alpha * \zeta_n|^p \mathrm{d}\xi = 0.
\end{aligned}$$

Similarly, one has

$$\lim_{n\to\infty} \|\nabla(X - X_n)\|^p_p = 0.$$

Moreover, noting that

$$\mathrm{div} X|_{U_\alpha} = X^k_\alpha \Gamma^i_{ki} + \partial_{\xi^\alpha_k} X^k_\alpha,$$

we have

$$\|[\partial_{\xi^\alpha_k} X^k_\alpha]^-\|_{L^\infty(U_\alpha)} \leqslant \|[\mathrm{div} X]^-\|_{L^\infty(M)} + C\|X\|_{L^\infty(TM)}.$$

Thus, by (3.9) we have

$$\begin{aligned}
\|[\mathrm{div}X_n]^-\|_{L^\infty(M)} &\leqslant \Big\| \sum_\alpha \Big(\psi_\alpha (X^k_{\alpha,n}\Gamma^i_{ki} + \partial_{\xi^k_\alpha} X^k_{\alpha,n}) + X^k_{\alpha,n}\partial_{\xi^\alpha_k}\psi_\alpha\Big)^- \Big\|_{L^\infty(M)} \\
&\leqslant \sum_\alpha \Big(\psi_\alpha (\|X^k_{\alpha,n}\Gamma^i_{ki}\|_{L^\infty(U_\alpha)} + \|[\partial_{\xi^k_\alpha} X^k_{\alpha,n}]^-\|_{L^\infty(U_\alpha)}) + \|X^k_{\alpha,n}\partial_{\xi^k_\alpha}\psi_\alpha\|_{L^\infty(U_\alpha)}\Big) \\
&\leqslant C(\|[\mathrm{div}X]^-\|_{L^\infty(M)} + \|X\|_{L^\infty(TM)}).
\end{aligned}$$

The proof is complete. □

4. Proof of Main Result

We first prove the following key estimation.

Lemma 4.1. *Let $x_t(x)$ and $\hat{x}_t(x)$ be two ν-almost everywhere stochastic flows of (1.1) corresponding to $(X_0, X_k, k = 1, \cdots, m)$ and $(\hat{X}_0, X_k, k = 1, \cdots, m)$, where*

$$X_0, \hat{X}_0 \in \mathbb{H}^p_1(TM) \text{ for some } p > 1 \text{ and } X_k \in C^2(TM), \quad k = 1, \cdots, m.$$

Then for any $\delta > 0$,

$$\mathbb{E}\int_M \log\left(\frac{\sup_{t\in[0,T]} \mathbf{d}^2(x_t(x), \hat{x}_t(x))}{\delta^2} + 1\right)\nu(\mathrm{d}x) \leqslant C_1 + \frac{C_2}{\delta}\|X_0 - \hat{X}_0\|_1,$$

where $C_1 = C\cdot(1 + K_{T,X_0,X_k} + K_{T,\hat{X}_0,X_k})(1 + \|X_0\|_{1,p})$ and $C_2 = C\cdot K_{T,\hat{X}_0,X_k}$. Here, K_{T,X_0,X_k} is from (2.1) and the constant C is independent of δ and $X_0, \hat{X}_0$.

Proof. Below, let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function satisfying

$$\chi(r) = r, \quad r \in [0, \lambda^4\varrho^2/4]; \quad \chi(r) = \lambda^4\varrho^2/2, \quad r \in [\lambda^4\varrho^2, \infty).$$

We define

$$f(x, y) := \chi(\mathbf{d}^2(x, y)).$$

Then by Lemma 3.2, $f \in C^\infty(M \times M)$ satisfies

$$f(x, y) \leqslant \mathbf{d}^2(x, y) \leqslant C_{\varrho,\lambda} f(x, y).$$

For the simplicity of notations, we write $z_t(x) := (x_t(x), \hat{x}_t(x))$. By Itô's formula, we have

$$\begin{aligned}
f(z_t(x)) &= \int_0^t [(X_0 f)_1 + (\hat{X}_0 f)_2](z_s(x))\mathrm{d}s + \int_0^t [(X_k f)_1 + (X_k f)_2](z_s(x)) \circ \mathrm{d}W^k_s \\
&= \int_0^t [(X_0 f)_1 + (\hat{X}_0 f)_2](z_s(x))\mathrm{d}s + \int_0^t [(X_k f)_1 + (X_k f)_2](z_s(x))\mathrm{d}W^k_s \\
&\quad + \frac{1}{2}\int_0^t [(X^2_k f)_{11} + (X^2_k f)_{21} + (X^2_k f)_{12} + (X^2_k f)_{22}](z_s(x))\mathrm{d}s,
\end{aligned}$$

where $(X_0 f)_1(x, y) = X_0(x) f(\cdot, y)$ and similarly for others. Using Itô's formula again, we further have

$$\begin{aligned}
\log\left(\frac{f(z_t(x))}{\delta^2} + 1\right) &= \int_0^t \frac{[(X_0 f)_1 + (\hat{X}_0 f)_2](z_s(x))}{f(z_s(x)) + \delta^2}\mathrm{d}s + \int_0^t \frac{[(X_k f)_1 + (X_k f)_2](z_s(x))}{f(z_s(x)) + \delta^2}\mathrm{d}W^k_s \\
&\quad + \frac{1}{2}\int_0^t \frac{[(X^2_k f)_{11} + (X^2_k f)_{21} + (X^2_k f)_{12} + (X^2_k f)_{22}](z_s(x))}{f(z_s(x)) + \delta^2}\mathrm{d}s \\
&\quad - \frac{1}{2}\int_0^t \frac{|[(X_k f)_1 + (X_k f)_2](z_s(x))|^2}{(f(z_s(x)) + \delta^2)^2}\mathrm{d}s \\
&=: I_1(t, x) + I_2(t, x) + I_3(t, x) + I_4(t, x).
\end{aligned}$$

Let us first treat $I_1(t,x)$. We write

$$\begin{aligned} I_1(t,x) &= \int_0^t \frac{[(X_0 f)_1 + (X_0 f)_2](z_s(x))}{f(z_s(x)) + \delta^2} \mathrm{d}s + \int_0^t \frac{[(\hat{X}_0 f)_2 - (X_0 f)_2](z_s(x))}{f(z_s(x)) + \delta^2} \mathrm{d}s \\ &=: I_{11}(t,x) + I_{12}(t,x). \end{aligned}$$

For a continuous real function $h(t)$, we write

$$h^*(T) := \sup_{t\in[0,T]} h(t).$$

By Lemma 3.6, we have

$$\begin{aligned} \mathbb{E}\int_M I_{11}^*(T,x)\nu(\mathrm{d}x) &\leq \mathbb{E}\int_M\int_0^T \frac{\chi'(\mathbf{d}^2(z_s(x)))\cdot|[(X_0\mathbf{d}^2)_1 + (X_0\mathbf{d}^2)_2](z_s(x))|}{\mathbf{d}^2(z_s(x)) + \delta^2} \mathrm{d}s\nu(\mathrm{d}x) \\ &\leq \mathbb{E}\int_0^T\int_M (1 + \mathcal{M}_\varrho|X_0|_1(x_s(x)) + \mathcal{M}_\varrho|X_0|_1(\hat{x}_s(x)))\nu(\mathrm{d}x)\mathrm{d}s \\ &\overset{(2.1)}{\leq} (K_{T,X_0,X_k} + K_{T,\hat{X}_0,X_k})\int_M (1 + \mathcal{M}_\varrho|X_0|_1(x))\nu(\mathrm{d}x) \\ &\overset{(3.3)}{\leq} (K_{T,X_0,X_k} + K_{T,\hat{X}_0,X_k})(1 + \|X_0\|_{1,p}). \end{aligned}$$

Noticing that

$$\begin{aligned} |(\hat{X}_0 f)_2 - (X_0 f)_2|(x,y) &= |\chi'(\mathbf{d}^2(x,y))\cdot((\hat{X}_0\mathbf{d}^2)_2 - (X_0\mathbf{d}^2)_2)(x,y)| \\ &\leqslant |\chi'(\mathbf{d}^2(x,y))|\cdot\mathbf{d}(x,y)\cdot|\hat{X}_0(y) - X_0(y)|_y, \end{aligned}$$

we similarly have

$$\begin{aligned} \mathbb{E}\int_M I_{12}^*(T,x)\nu(\mathrm{d}x) &\leq \frac{1}{\delta}\mathbb{E}\int_0^T\int_M |\hat{X}_0(\hat{x}_s(x)) - X_0(\hat{x}_s(x))|_{\hat{x}_s(x)}\nu(\mathrm{d}x)\mathrm{d}s \\ &\leq \frac{K_{T,\hat{X}_0,X_k}}{\delta}\int_M |X_0 - \hat{X}_0|_x\nu(\mathrm{d}x). \end{aligned}$$

For $I_2(t,x)$, by BDG's inequality and Lemma 3.6, we have

$$\mathbb{E}\int_M I_2^*(T,x)\nu(\mathrm{d}x) \leq \int_M \mathbb{E}\left(\int_0^T \left|\frac{[(X_k f)_1 + (X_k f)_2](z_s(x))}{f(z_s(x)) + \delta^2}\right|^2 \mathrm{d}s\right)^{1/2} \nu(\mathrm{d}x) \leqslant C,$$

where the constant C is independent of δ and may depend on X_k. Similarly, by Lemma 3.7, we also have

$$\mathbb{E}\int_M I_3^*(T,x)\nu(\mathrm{d}x) \leqslant C.$$

Since $I_4(t,x)$ is negative, this term can be dropped. Combining the above calculations, we obtain the desired estimate. □

We also recall the following results for later use (cf. [16]).

Lemma 4.2. *Let $x_n(\omega,x):\Omega\times M\to M, n\in\mathbb{N}$ be a family of measurable mappings. Suppose that for P-almost all $\omega\in\Omega$, $\nu\circ x_n(\omega,\cdot)\ll\nu$ and the density $\beta_n(\omega,x)$ satisfies*

$$\sup_n \sup_{x\in M} \mathbb{E}|\beta_n(x)|^2 \leqslant C_1. \tag{4.1}$$

If for $(P\times\nu)$-almost all $(\omega,x)\in\Omega\times M$, $x_n(\omega,x)\to x_0(\omega,x)$ as $n\to\infty$, then for P-almost all $\omega\in\Omega$, $\nu\circ x_0(\omega,\cdot)\ll\nu$ and the density β also satisfies

$$\sup_{x\in M} \mathbb{E}|\beta(x)|^2 \leqslant C_1. \tag{4.2}$$

Moreover, let $(f_n)_{n\in\mathbb{N}}$ be a family of uniformly bounded and measurable functions on M. If f_n converges to some f in $L^1(M)$, then

$$\lim_{n\to\infty}\mathbb{E}\int_M |f_n(x_n(x)) - f(x_0(x))|\nu(\mathrm{d}x) = 0. \tag{4.3}$$

Lemma 4.3. *Let $\mathcal{T},\hat{\mathcal{T}} : M \to M$ be two measurable transformations. Let $\mathscr{C}$ be a countable and dense subset of $C(M)$. Let $\rho \in L^1(M)$ be a positive measurable function. Assume that for any $f, g \in \mathscr{C}$,*

$$\int_M f(\hat{\mathcal{T}}(x))\cdot g(x)\nu(\mathrm{d}x) = \int_M f(x)\cdot g(\mathcal{T}(x))\cdot\rho(x)\nu(\mathrm{d}x).$$

Then $\mathcal{T}$ admits a measurable invertible $\hat{\mathcal{T}}$, i.e., $\mathcal{T}^{-1}(x) = \hat{\mathcal{T}}(x)$ a.e.. Moreover,

$$\nu\circ\mathcal{T}^{-1} = \rho\nu, \quad \nu\circ\mathcal{T} = \rho^{-1}(\mathcal{T}^{-1})\nu.$$

Proposition 4.4. *Consider SDE (1.1) with $X_k \in C^2(TM)$, $k = 0, 1, \cdots, m$. Let $x_t(x)$ be the unique stochastic homeomorphism flow associated with SDE (1.1). Then*

$$\nu\circ x_t^{-1}(\mathrm{d}x) \sim \nu(\mathrm{d}x), \quad \nu\circ x_t(\mathrm{d}x) \sim \nu(\mathrm{d}x)$$

and

$$\nu\circ x_t^{-1}(\mathrm{d}x) = \exp\left\{\int_0^t \mathrm{div}X_0(x_s(x))\mathrm{d}s + \int_0^t \mathrm{div}X_k(x_s(x))\circ\mathrm{d}W_s^k\right\}\nu(\mathrm{d}x). \tag{4.4}$$

Moreover, for any $q \geqslant 1$

$$\mathbb{E}\left|\frac{\nu\circ x_t^{-1}(\mathrm{d}x)}{\nu(\mathrm{d}x)}\right|^q \leqslant \exp\left\{C_q T(\|[\mathrm{div}X_0]^+\|_\infty + \|\mathrm{div}X_k\|_\infty^2 + \|X_k\mathrm{div}X_k\|_\infty)\right\} \tag{4.5}$$

and

$$\mathbb{E}\left|\frac{\nu\circ x_t(\mathrm{d}x)}{\nu(\mathrm{d}x)}\right|^q \leqslant \exp\left\{C_q T(\|[\mathrm{div}X_0]^-\|_\infty + \|\mathrm{div}X_k\|_\infty^2 + \|X_k\mathrm{div}X_k\|_\infty)\right\}. \tag{4.6}$$

Proof. We sketch the proof. Let $W_{n,t}$ be the linearized approximation of W_t. Consider the following ODE on M:

$$\mathrm{d}x_{n,t}(x) = X_0(x_{n,t}(x))\mathrm{d}t + X_k(x_{n,t}(x))\dot{W}_{n,t}^k\mathrm{d}t.$$

It is a well known fact that

$$\nu\circ x_{n,t}^{-1}(\mathrm{d}x) = \exp\left\{\int_0^t \mathrm{div}X_0(x_{n,s}(x))\mathrm{d}s + \int_0^t \mathrm{div}X_k(x_{n,s}(x))\dot{W}_{n,s}^k\mathrm{d}s\right\}\nu(\mathrm{d}x)$$

By the limit theorem (cf. [10], [15], [13]), the desired formula (4.4) then follows.

Note that

$$\int_0^t \mathrm{div}X_k(x_s(x))\circ\mathrm{d}W_s^k = \int_0^t \mathrm{div}X_k(x_s(x))\mathrm{d}W_s^k + \frac{1}{2}\int_0^t X_k\mathrm{div}X_k(x_s(x))\mathrm{d}s$$

and

$$t\mapsto\exp\left\{q\int_0^t \mathrm{div}X_k(x_s(x))\mathrm{d}W_s^k - \frac{q^2}{2}\int_0^t |\mathrm{div}X_k|^2(x_s(x))\mathrm{d}s\right\}$$

is an exponential martingale. It is easy to see that (4.5) holds. (4.6) can be proved similarly (cf. [16]). □

We now prove the following result.

Theorem 4.5. *Assume that $X_0 \in \mathbb{H}^p_1(TM) \cap L^\infty(TM)$ for some $p > 1$ satisfies*

$$[\mathrm{div} X_0]^- \in L^\infty(M),$$

and for each $k = 1, \cdots, m$, $X_k \in C^2(TM)$. Then there exists a unique ν-almost everywhere stochastic flows $\{x_t(x), x \in M\}_{t\geqslant 0}$ associated with SDE (1.1) in the sense of Definition 2.1.

Proof. Let $X_{0,n} \in C^\infty(TM)$ be defined as in (3.10). Let $x_{n,t}(x)$ solve the following Stratonovich's SDE on M:

$$\mathrm{d}x_{n,t}(x) = X_{0,n}(x_{n,t}(x))\mathrm{d}t + X_k(x_{n,t}(x)) \circ \mathrm{d}W^k_t, \quad x_{n,0} = x.$$

Then $x \mapsto x_{n,t}(x), t \geqslant 0$ defines a stochastic homeomorphism flow over M. Moreover, by Proposition 4.4

$$(\nu \circ x_{n,t})(\mathrm{d}x) = \beta_{n,t}(x)\nu(\mathrm{d}x),$$

where $\beta_{n,t}(x)$ satisfies by (4.6) and (3.11), that for any $q \geqslant 1$,

$$\sup_{n\in\mathbb{N}} \sup_{(t,x)\in[0,T]\times M} \mathbb{E}|\beta_{n,t}(x)|^q < +\infty. \tag{4.7}$$

Let us set

$$\Phi_{n,m}(x) := \sup_{t\in[0,T]} \mathbf{d}^2(x_{n,t}(x), x_{m,t}(x))$$

and

$$A^\delta_{n,m}(x) := \log\left(\frac{\Phi_{n,m}(x)}{\delta} + 1\right).$$

If we choose

$$\delta = \delta_{n,m} = \|X_{0,n} - X_{0,m}\|_1,$$

then by Lemma 4.1 and (4.7), we have

$$\sup_{n,m} \mathbb{E} \int_M A^{\delta_{n,m}}_{n,m}(x)\nu(\mathrm{d}x) \leqslant C_0.$$

Thus, by Chebyshev's inequality, we have for any $R > 0$,

$$\begin{aligned}
\mathbb{E}\int_M \Phi_{n,m}(x)\nu(\mathrm{d}x) &= \mathbb{E}\int_M \Phi_{n,m}(x)\cdot 1_{\{A^{\delta_{n,m}}_{n,m}(x)>R\}}\nu(\mathrm{d}x) + \mathbb{E}\int_M \Phi_{n,m}(x)\cdot 1_{\{A^{\delta_{n,m}}_{n,m}(x)\leqslant R\}}\nu(\mathrm{d}x)\\
&\leqslant \frac{(\mathrm{diam}(M))^2\cdot C_0}{R} + \delta_{n,m}\cdot(e^R - 1)\cdot\nu(M)\\
&= \frac{(\mathrm{diam}(M))^2\cdot C_0}{R} + \|X_{0,n} - X_{0,m}\|_1\cdot(e^R-1)\cdot\nu(M),
\end{aligned}$$

where $\mathrm{diam}(M) := \sup_{x,y\in M}\mathbf{d}(x,y)$. From this, by Proposition 3.9, we then obtain that

$$\lim_{n,m\to\infty}\mathbb{E}\int_M \sup_{t\in[0,T]} \mathbf{d}^2(x_{n,t}(x), x_{m,t}(x))\nu(\mathrm{d}x) = \lim_{n,m\to\infty}\mathbb{E}\int_M \Phi_{n,m}(x)\nu(\mathrm{d}x) = 0.$$

Hence, for ν-almost all $x \in M$, there exists a continuous $(\mathcal{F}_t)$-adapted process $x_t(x)$ such that

$$\lim_{n\to\infty}\mathbb{E}\int_M \sup_{t\in[0,T]} \mathbf{d}^2(x_{n,t}(x), x_t(x))\nu(\mathrm{d}x) = 0. \tag{4.8}$$

By Lemma 4.2 and (4.7), one finds that $x_t(x)$ satisfies **(A)** and **(B)** of Definition 2.1. The uniqueness is a direct consequence of Lemma 4.1. □

We are now in a position to give

Proof of Theorem 2.5:

Following the proof of Theorem 4.5, we only need to check **(C)** of Definition 2.1. Fix a $T > 0$ and let

$$\rho_n := \exp\left\{\int_0^T \mathrm{div}X_{0,n}(x_{n,s}(x))\mathrm{d}s + \int_0^T \mathrm{div}X_k(x_{n,s}(x)) \circ \mathrm{d}W_s^k\right\}.$$

By (4.5), we have for any $q \geqslant 1$,

$$\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} \mathbb{E}|\rho_n(x)|^q < +\infty. \tag{4.9}$$

In view of (4.7) and (4.8), by Lemma 4.2, we have

$$\lim_{n\to\infty} \mathbb{E}\int_0^T\int_M |\mathrm{div}X_{0,n}(x_{n,s}(x)) - \mathrm{div}X_0(x_s(x))|\nu(\mathrm{d}x)\mathrm{d}s = 0,$$

$$\lim_{n\to\infty} \mathbb{E}\int_M \left|\int_0^T (\mathrm{div}X_k(x_{n,s}(x)) - \mathrm{div}X_k(x_s(x))) \circ \mathrm{d}W_s^k\right| \nu(\mathrm{d}x) = 0.$$

So, there is a subsequence still denoted by n such that for almost all (ω, x),

$$\lim_{n\to\infty} \rho_n(\omega, x) = \rho_T(\omega, x), \tag{4.10}$$

where $\rho_T(x)$ is defined by (2.2). By (4.9) and (4.10), we further have for any $q \geqslant 1$,

$$\lim_{n\to\infty} \mathbb{E}\int_M |\rho_n(x) - \rho_T(x)|^q \nu(\mathrm{d}x) = 0. \tag{4.11}$$

Now, let $y_n(x)$ solve the following SDE

$$\mathrm{d}y_{n,t}(x) = -X_{0,n}(y_{n,t}(x))\mathrm{d}t + X_k(y_{n,t}(x)) \circ \mathrm{d}W_t^{T,k}, \ \ y_n|_{t=0} = x,$$

where $W_t^T := W_{T-t} - W_T$. As in the proof of Theorem 4.5, there exists a continuous $(\mathcal{F}_t)$-adapted process $y_t(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}\int_M \sup_{t\in[0,T]} \mathbf{d}(y_{n,t}(x), y_t(x))^2 \nu(\mathrm{d}x) = 0. \tag{4.12}$$

It is well known that

$$x_{n,T}^{-1}(x) = y_{n,T}(x).$$

Thus, for any $f, g \in C(M)$, we have

$$\int_M f(y_{n,T}(\omega, x)) \cdot g(x)\nu(\mathrm{d}x) = \int_M f(x) \cdot g(x_{n,T}(\omega, x)) \cdot \rho_n(\omega, x)\nu(\mathrm{d}x), \ \ P - a.s. \tag{4.13}$$

Let $\mathscr{C}$ be a countable and dense subset of $C(M)$. By (4.8), (4.11) and (4.12), if necessary, extracting a subsequence and then taking limits $n \to \infty$ in $L^1(\Omega)$ for both sides of (4.13), we get that for all $f, g \in \mathscr{C} \subset C(M)$ and P-almost all $\omega \in \Omega$,

$$\int_M f(y_T(\omega, x)) \cdot g(x)\nu(\mathrm{d}x) = \int_M f(x) \cdot g(x_T(\omega, x)) \cdot \rho_T(\omega, x)\nu(\mathrm{d}x). \tag{4.14}$$

Since $\mathscr{C}$ is countable, one may find a common null set $\Omega' \subset \Omega$ such that (4.14) holds for all $\omega \notin \Omega'$ and $f, g \in \mathscr{C}$. Thus, by Lemma 4.3, one sees that **(C)** of Definition 2.1 holds.

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